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Notices of the Aëronautical Society of Great Britain.

Members' Subscriptions.

Those Members who have not yet paid their subscriptions for the current year are reminded that these fall due on the 1st January each year.

Wilbur Wright Lecture, 1916.

The Fourth Wilbur Wright Memorial Lecture will be delivered on Tuesday, June 6th next, at three p.m., in the Theatre of the Royal Society of Arts, John Street, Adelphi, W.C.

The title is "The Life and Work of Wilbur Wright," and the lecturer, Mr. Griffith Brewer, A.F.Aë.S.

The Right Hon. the Lord Montagu of Beaulieu will preside.

Tickets will shortly be available for distribution from the Offices of the Society, 11, Adam Street, Strand, W.C., to which address applications should be made.

Edward Busk Studentship in Aeronautics.

Applications will shortly be invited for the above Aeronautical Research Studentship. The approved scheme provides that the award and control of the Studentship are to be in the hands of a Committee of whom three members are appointed biennially by the Aëronautical Society, and three by King's College, Cambridge (Edward Busk's old College), with two of the Trustees of the Fund.

The remuneration of the Student will be at a rate not exceeding £150 per annum for a period of one year, with an extension in special circumstances to two years. The Student will be elected by the Committee without examination and shall not have completed his 25th year on the 1st day of October preceding the election.

The Student will be required to make reports to the Committee on his work from time to time, and at the close to present a full report to the Aëronautical Society, the University of Cambridge and the Trustees, and he may be required to deliver a lecture or lectures embodying the substance of the Report.

The intention of the founders of the Studentship is that it should be primarily for the purpose of encouraging and helping young aeronautical engineers to make a good start on their career.

Inquiries should be addressed to the Secretary, Aëronautical Society, 11, Adam Street, Strand, W.C., from which address application forms will shortly be procurable.

Edward Busk Studentship Committee.

Two of the Society's three representatives on the above Committee have been appointed, viz. :—Mr. Horace Darwin, F.R.S., F.Aë.S. and Dr. R. T. Glazebrook, C.B., F.R.S., F.Aë.S. The name of the third representative will shortly be announced.

Annual General Meeting, 1916.

The Annual General Meeting of the Society will take place, as last year, immediately preceding the Wilbur Wright Lecture.

AGENDA.

To receive and approve the Report of the Council on the state of the Society, and the Balance Sheet of Aërial Science, Limited.

To discuss and determine such questions as may be proposed by the voters relating to the affairs of the Society, and to fill the vacancies of the Council for the ensuing year. Any voter desirous of proposing any subject for discussion at the Annual General Meeting shall give notice in writing to the Secretary, which shall be received by him by noon on 23rd May, 1916.

The retiring Members of Council are :—

HARRIS BOOTH,	Squadron Commander F. K. MCCLEAN,
J. H. LEDEBOER,	Major-Gen. R. M. RUCK, C.B., R.E.,
Lieut. A. R. LOW, R.N.V.R.,	Lieut.-Col. F. H. SYKES, C.M.G.,
Dr. T. E. STANTON, F.R.S.,	Dr. R. MULLINEUX WALMSLEY,

who are eligible for re-election.

Nominations of candidates for election to the Council shall be signed by the voters proposing them (two voters and no more), and must be received by the Secretary by noon on 16th May, 1916, with an intimation in writing by the voters nominated that they are willing to serve.

The Council will also ask for approval of their action in postponing, on account of the war, the date of the Annual General Meeting.

Elections.

The following elections to the Society, some of which were, by oversight, not previously published, are notified :—

Members : J. C. G. SYKES, C.B., dating from October, 1915.
ALEXANDER WADDELL, „ October, 1915.
J. A. WHITEHEAD, „ October, 1915.
Lieut. JOHN SOWREY, R.F.C. „ January, 1916.

Associate Members : FRANCIS WARD, dating from October, 1915.
JOSEPH HOOLEY, „ January, 1916.
H. SKELTON, „ March, 1916.

Students : L. A. C. DE LA GARDE, dating from November, 1915.
S. L. GREEN, „ February, 1916.
P. C. THORNTON, „ February, 1916.

FOURIER'S THEOREM IN AERODYNAMICS.

BY PROF. G. H. BRYAN,

WITH TABLES DRAWN UP BY CARADOG WILLIAMS, B.SC.

PART I.—PLANE SURFACES.

1. In "Practical Aerodynamics," Major B. Baden-Powell quotes the following formulæ for the pressure of the air on a moving plane when expressed as a function of the angle of attack α, taking the pressure as unity when $\alpha = 90°$.

I. Duchemin

$$\frac{2 \sin \alpha}{1 + \sin^2 \alpha}$$

II. Rayleigh (hydrodynamical)

$$\frac{\pi \sin \alpha}{4 + \pi \sin \alpha}$$

II*a*. Gerlach

$$\frac{(4 + \pi) \sin \alpha}{4 + \pi \sin \alpha}$$

III. Renard

$$\sin \alpha \, [2 - (\alpha - 1) \sin^2 \alpha]$$

IV. Soreau

$$\sin \alpha \left(1 + \frac{1}{1 + \tan^2 \alpha} \right)$$

V. Eiffel, 1907,

$$\begin{cases} \alpha/30 \text{ from } \alpha = 0 \text{ to } \alpha = 30° \\ 1 \text{ from } \alpha = 30° \text{ to } \alpha = 90° \end{cases}$$

VI. Another formula frequently quoted is

$$\begin{matrix} 2 \sin \alpha \text{ from } \alpha = 0 \text{ to } \alpha = 30° \\ 1 \text{ from } \alpha = 30° \text{ to } \alpha = 90° \end{matrix}$$

and a more complicated formula, depending on the aspect ratio of the plane, has been proposed by Soreau.

The use of Fourier's series affords a convenient means for standardising and comparing the results both of these formulæ and of experimental investigations. It might naturally have been supposed that this method would be the first to suggest itself to the earlier writers on the subject, but the fact that results are not given in the form of such a series in the books which are most read indicates that this method has been wholly or partially overlooked, or at any rate has not received so much attention as it appears to deserve.

The object of the present paper is to illustrate the application of Fourier's theorem to the various aerodynamical magnitudes which are functions of the angle of attack of a moving plane, viz., resultant thrust, lift, drag, co-ordinate

of centre of pressure and moment of resultant thrust about the centre of area. The formulæ quoted above afford a sufficient variety of examples of the application of the method to empirical formulæ, while the tables contained in Eiffel's "Resistance of the Air and Aviation," translated by Hunsaker, give a sufficient diversity of experimental data to exemplify the methods of determining the Fourier co-efficients from tabulated statistics. To calculate the expansions for every series that has been proposed or for every collection of experimental results that has been tabulated would of course be out of the question, and it is therefore intended that the data here given should be regarded as illustrating the methods and the general character of the results rather than as leading to anything final. As fresh experiments are made and fresh formulæ proposed, it should be easy to represent them in the same form.

The advantages of the Fourier analysis are

(1) The *standardised* form in which different results can be expressed, the only differences being in the values of the numerical coefficients.

(2) The *elasticity* of the formulæ which can be adapted to functions of almost any kind whatever.

(3) The possibility of obtaining *approximate* formulæ by retaining only a few terms of the series, or of obtaining closer approximations in some cases by the addition of further terms.

2. *Duchemin's formula* can be most easily expanded by means of De Moivre's theorem. Putting $x = \cos\alpha + i\sin\alpha$ we find

$$\frac{1}{1+\sin^2\alpha} = \frac{1}{1+\frac{(x-x^{-1})^2}{4x^2}} = \frac{-4}{-6+x^2+x^{-2}} = \frac{-4x^2}{x^4-6x^2+1}$$

$$= \frac{4}{q-p}\left\{\frac{p}{x^2-p} - \frac{q}{x^2-q}\right\}$$

where

$$p = 3-2\sqrt{2},\ q = 3+2\sqrt{2},\ q-p = 4\sqrt{2},\ qp = 1.$$

Expanding the fractions in ascending powers of p or $1/q$ we obtain

$$\sqrt{2}\left\{\tfrac{1}{2} + p\cos 2\alpha + p^2\cos 4\alpha + p^3\cos 6\alpha + \ldots\right\}$$

Multiplying by $2\sin\alpha$ and using the usual trigonometric formula we obtain

$$\frac{2\sin\alpha}{1+\sin^2\alpha} = (4-2\sqrt{2})(\sin\alpha + p\sin 3\alpha + p^2\sin 5\alpha, \text{ etc.}) \quad . \qquad (1)$$

where

$$p = 3-2\sqrt{2} = 0.1716$$

This series, it will be seen, converges very rapidly and is therefore convenient for calculation.

3. *Soreau's formula* as quoted above is still simpler as it gives

$$\sin\alpha\left(1+\frac{1}{1+\tan^2\alpha}\right) = 1.25\sin\alpha + 0.25\sin 3\alpha \quad . \quad . \qquad (2)$$

Both the above formulæ hold good for all values of the angle α. If α is negative and between 0 and -180° the pressure is on the opposite face of the plane and therefore negative. Also both expressions become equal to unity

when $\alpha = 90°$ and therefore they correctly represent the pressure ratio $P(\alpha) : P(90°)$.

Rayleigh's formula II. can only hold good between 0 and 180° and it is obvious from symmetry that Renard's formula III. cannot hold good beyond 90°, as it would make $P(180° - \alpha)$ unequal to $P(\alpha)$, where P is the pressure.

Rayleigh's expression does not become equal to unity when $\alpha = 90°$, but Gerlach's modification (III.a) does so. In the case of Renard's formula we must divide by $3 - \frac{1}{2}\pi$ to obtain the pressure ratio $P(\alpha) : P(90°)$.

In all the formulæ, except I. and IV., the expansion must be performed by the usual Fourier methods.

4. *Digression on Fourier Series.*—The correct form of Fourier's theorem states that if $f(\alpha)$ is a function of the angle α which is given for all values of α from 0 to 360°, and which outside these limits satisfies the condition $f(n.360° + \alpha) = f(\alpha)$ then $f(\alpha)$ can be expanded in a series of sines and cosines of multiples of α, and the coefficients of $\cos n\alpha$ and $\sin n\alpha$ can be obtained by multiplying by $\cos n\alpha$ and $\sin n\alpha$ respectively and integrating between the limits 0 and 360° (or 0 and 2π since the integration usually has to be effected in radians).

If the value of $f(\alpha)$ is only given for angles in the first quadrant it would be possible to find any number of different series representing the function, by assigning arbitrary values to $f(\alpha)$ in the remaining quadrants, but the following forms are the most convenient.

$$\text{A} \quad f(\alpha) = f(180° - \alpha) = f(180° + \alpha) = f(-\alpha)$$

Even cosine series

$$\text{B} \quad f(\alpha) = \begin{matrix} a_0 + a_2 \cos 2\alpha + a_4 \cos 4\alpha + \ldots \\ -f(180° - \alpha) = -f(180° + \alpha) = f(-\alpha) \end{matrix}$$

Odd cosine series

$$\text{C} \quad f(\alpha) = \begin{matrix} a_1 \cos \alpha + a_3 \cos 3\alpha + a_5 \cos 5\alpha + \ldots \\ -f(180° - \alpha) = +f(180° + \alpha) = -f(-\alpha) \end{matrix}$$

Even sine series

$$\text{D} \quad f(\alpha) = \begin{matrix} b_0 \sin 2\alpha + b_2 \sin 4\alpha + b_4 \sin 6\alpha + \ldots \\ f(180° - \alpha) = -f(180° + \alpha) = -f(-\alpha) \end{matrix}$$

Odd sine series

$$b_1 \sin \alpha + b_3 \sin 3\alpha + b_5 \sin 5\alpha + \ldots$$

Having regard to the conditions of symmetry, we see that if the formulæ are to hold good for angles of any magnitude in the case of a plane symmetrical lamina such as a rectangle—

1. The ratio $P(\alpha) : P(90°)$ must be expressed in the form of a series of sines of odd multiples of α, or as we say for shortness, an odd sine series, and it will be seen that this form is uniform with the expansions of Duchemin's and Soreau's formulæ given above. We shall call this expansion

$$P_1 \sin \alpha + P_3 \sin 3\alpha + P_5 \sin 5\alpha + \ldots \quad . \quad . \quad (3)$$

2. The distance of the centre of pressure from the centre of figure must be expressed as an odd cosine series, say,

$$C_1 \cos \alpha + C_3 \cos 3\alpha + C_5 \cos 5\alpha + \ldots \quad . \quad . \quad (4)$$

3. The moment of the resultant pressure about the centre of figure must be expressed as an even sine series, say,

$$G_2 \sin 2\alpha + G_4 \sin 4\alpha + G_6 \sin 6\alpha. \quad . \quad . \quad . \quad (5)$$

5. *Modification for Small Angles—Cosine Series.*—If we use the formula

$$f(\alpha) = P_1 \sin \alpha + P_3 \sin 3\alpha + P_5 \sin 5\alpha + \ldots$$

for the ratio $f(\alpha) = P(\alpha) : P(90°)$ it will be seen that when α is small the ratio of $f(\alpha)$ to $\sin \alpha$ approximates to the value

$$P_1 + 3P_3 + 5P_5 + \ldots$$

Although this series is really convergent in the cases which occur in aeronautics it is obviously very inconvenient for calculation as it may converge very slowly. To meet this difficulty it is sometimes better to expand $f(\alpha)/\sin \alpha$ in the form of an even cosine series so that, say,

$$\begin{aligned} f(\alpha) &= \sin \alpha \times (Q_0 + Q_2 \cos 2\alpha + Q_4 \cos 4\alpha + \ldots) \\ &= Q_0 \sin \alpha + \tfrac{1}{2}Q_2 (\sin 3\alpha - \sin \alpha) + \tfrac{1}{2}Q_4 (\sin 5\alpha - \sin 3\alpha) + \ldots \end{aligned}$$

Thus

$$P_1 = Q_0 - \tfrac{1}{2}Q_2, \; P_3 = \tfrac{1}{2}(Q_2 - Q_4), \; P_5 = \tfrac{1}{2}(Q_4 - Q_6). \quad . \quad (6)$$

Knowing the cosine series for $f(\alpha)/\sin \alpha$ the corresponding sine series for $f(\alpha)$ is thus easily deduced, but the converse is not possible unless the value of Q_0 is known. Of course we have, however,

$$Q_0 = P_1 + P_3 + P_5 + \ldots$$

which may be used if the series is sufficiently convergent or if a small number of terms give the necessary degree of approximation.

If, however, Q_0 has been calculated independently the remaining coefficients of the Q series can easily be deduced from the P series.

6. *Expansion of Eiffel's Form (V.)*—By multiplying and integrating between 0 and $\frac{1}{2}\pi$ we get, using circular measure and writing x for α,

$$\frac{\pi}{4} P_n = \int_0^{\pi/6} \frac{6}{\pi} x \sin nx \, dx + \int_{\pi/6}^{\pi/2} \sin nx \, dx$$

$$P_n = \frac{24}{\pi^2 n^2} \sin \frac{n\pi}{6}$$

Putting $n = 1, 3, 5, 7$, etc., the expansion gives

$$f(\alpha) = \frac{12}{\pi^2} \left(\frac{\sin \alpha}{1^2} + \frac{2 \sin 3\alpha}{3^2} + \frac{\sin 5\alpha}{5^2} - \frac{\sin 7\alpha}{7^2} - \frac{2 \sin 9\alpha}{9^2} - \frac{\sin 11\alpha}{11^2} + \ldots \right) \quad (7)$$

the coefficients containing factors which recur in the order 1, 2, 1, —1, —2, —1, 1, 2, 1. It will be seen that the coefficients change sign after the first three, which is not the case in Duchemin's formula.

For the corresponding form (VI.) we find

$$\frac{\pi}{4} P_n = \frac{\sin (n-1)\pi/6}{n(n-1)} + \frac{\sin (n+1)\pi/6}{n(n+1)}$$

$$P_1 = \frac{2}{3} + \frac{\sqrt{3}}{\pi}$$

and the series becomes

$$\frac{2\sin\alpha}{3} + \frac{2\sqrt{3}}{\pi}\left\{\frac{\sin\alpha}{1.2} + \frac{2\sin 3\alpha}{2.4} + \frac{\sin 5\alpha}{4.5} - \frac{\sin 7\alpha}{7.8} - \frac{2\sin 9\alpha}{8.10} - \frac{\sin 11\alpha}{10.11} + \frac{\sin 13\alpha}{13.14} \ldots\right\} \quad . \quad . \quad (8)$$

while if it be required to use the cosine series for $f(\alpha)/\sin\alpha$ we have

$$Q_0 = \frac{2}{3} + \frac{2\log_{10}(2+\sqrt{3})}{\log_{10} e}$$

7. *Renard's formula*, III., when expanded in the same way leads to

$$\sin\alpha\left\{2-(\alpha-1)\sin^2\alpha\right\} = \left(\frac{11}{4} - \frac{3\pi}{16} - \frac{1}{\pi}\right)\sin\alpha - \left(\frac{1}{4} - \frac{\pi}{16}\right)\sin 3\alpha + \frac{1}{\pi}\left\{\left(\frac{3}{2^2}+\frac{1}{6^2}\right)\sin 3\alpha - \left(\frac{3}{6^2}+\frac{1}{2^2}\right)\sin 5\alpha + \left(\frac{3}{6^2}+\frac{1}{10^2}\right)\sin 7\alpha - \left(\frac{3}{10^2}+\frac{1}{6^2}\right)\sin 9\alpha + \left(\frac{3}{10^2}+\frac{1}{14^2}\right)\sin 11\alpha, \text{ etc.}\right\} \quad . \quad (9)$$

while the constant term of the cosine series for $\left\{2-(\alpha-1)\sin^2\alpha\right\}$ is equal to

$$\frac{5}{2} - \frac{\pi}{8} - \frac{1}{2\pi}$$

8. *Rayleigh's and Gerlach's formulæ.*—The determination of the coefficients in the expansion in this case depends on the evaluations of integrals of the form

$$\int_0^{\pi/2} \frac{\cos nx\,dx}{\sin x + k} \text{ or } \int_0^{\pi/2} \frac{\sin nx\,dx}{\sin x + k} \text{ where } p = \frac{4}{\pi}$$

which we will denote by I_n and J_n, the choice depending on whether it is decided to use the sine or cosine series. The evaluation of I_0, I_1 and J_1 is discussed in all elementary text books (for example, Edwards's Integral Calculus for Beginners, §§74, 75, gives all that is necessary). For the subsequent terms of the series we may use a formula of reduction based on the formulæ

$$\cos nx + \cos(n-4)x = 2\cos(n-2)x\,(1-2\sin^2 x)$$
$$\sin nx + \sin(n-4)x = 2\sin(n-2)x\,(1-2\sin^2 x)$$

Now $1-2\sin^2 x = 1-2k^2-2(\sin x + k)(\sin x - k)$ and we easily deduce

$$I_n + (4k^2-2)I_{n-2} + I_{n-4} = -4\int_0^{\pi/2}(\sin x - k)\cos(n-2)x\,dx \quad . \quad (10)$$

and a similar form for the J_n integrals. The only objection to this method is that since $4k^2-2$ is roughly equal to 4.4, a small error in the value of one term will produce increasing errors in the subsequent terms, and if the calculations are carried far enough the effects of such an error will be to make the coefficients ultimately divergent as Mr. Creak has verified by actual calculation. For the

calculations required in aeronautical applications all that is necessary is to calculate the first terms to a few extra decimal places in order to get the subsequent ones correct. In some of Mr. Creak's calculations an alternative formula, obtained by the use of De Moivre's Theorem, has been used, and is free from this objection.

The numerical coefficients in the various expansions have been evaluated by Mr. T. G. Creak, M.A., of Llanberis, and are given in Table A. While the results represent the expansions of the functions in question, it is quite evident that for aeronautical purposes a few figures only should be retained.

9. *Calculations from experimental data.*—The methods of evaluating the coefficients in a Fourier expansion from statistical data have been fairly well investigated by mathematicians and others. In most cases it is assumed that the data have been tabulated at equidistant intervals, say, at intervals of 8° or 10° and in such cases the methods are well known. But in aeronautical experiments, such as those of Eiffel, the intervals are different in different parts of the scale; for example, in one of Eiffel's experiments with a square plate, the data are tabulated for angles of 5, 10, 20, 30, 35, 38, 40, 45, 60, 75, 90 degrees. Moreover, it is very important that the method adopted should be as simple as possible, because if it is to be of any use it must be capable of being employed by practical experts who have no time for making elaborate calculations. Also it would be quite unnecessary and indeed unadvisable to carry the figures to a high degree of approximation, and, indeed, it will certainly be necessary to cut down the superfluous places of decimals in the calculations tabulated in illustration of this paper.

It thus becomes necessary to resort to approximate methods of integration, and after examining a number of methods I have come to the conclusion that a generalisation of the trapezoidal rule is the best method to adopt. In other words, when the values of $f(\alpha)$ have been plotted in a graph, the quantity to be expanded is taken to be represented by the polygonal outline formed by joining the points up by straight lines instead of a continuous curve passing through the points. The formula required is obtained as follows:—

Taking

$$y = f(\alpha) = P_1 \sin \alpha + P_3 \sin 3\alpha + P_5 \sin 5\alpha$$

we have on multiplying and integrating, and then integrating by parts and replacing α by x,*

$$\frac{\pi}{4} P_n = \int_0^{\pi/2} f(x) \sin nx \, dx = -\left[\frac{f(x) \cos nx}{n}\right]_0^{\pi/2} + \int_0^{\pi/2} \frac{f^1(x) \cos nx}{n} \, dx \quad (11)$$

In the case of pressure on a plane lamina $f(0) = 0$, also $\cos n\pi/2 = 0$ since n is odd, the term in square brackets therefore vanishes.

For the polygonal outline if (x_r, y_r) and (x_{r+1}, y_{r+1}) are two consecutive vertices we have for this element

$$f^1(x) = \frac{y_{r+1} - y_r}{x_{r+1} - x_r} = \left[\frac{\Delta y}{\Delta x}\right]_r^{r+1}$$

* x is taken to denote the independent variable in the integration formulæ while α denotes the angle of which the function is expanded.

say, and is constant in the interval. This gives

$$\frac{\pi}{4} P_n = \Sigma \left[\frac{\Delta y}{\Delta x}\right]_r^{r+1} \frac{\sin nx_{r+1} - \sin nx_r}{n^2}$$

$$= \frac{1}{n^2} \Sigma \sin nx_r \left\{ \left[\frac{\Delta y}{\Delta x}\right]_{r-1}^{r} - \left[\frac{\Delta y}{\Delta x}\right]_r^{r+1} \right\}$$

which we may write

$$= -\frac{1}{n^2} \Sigma \sin nx \, \Delta \frac{\Delta y}{\Delta x}$$

In this formula angles are measured in radians. In practical work it is necessary to measure Δx in degrees, and when this is done we find

$$P_n = \frac{4 \times 180}{\pi^2 n^2} \Sigma \sin nx_r \left\{ \left[\frac{\Delta y}{\Delta x}\right]_{r-1}^{r} - \left[\frac{\Delta y}{\Delta x}\right]_r^{r+1} \right\} \quad . \quad (12)$$

The values of the logarithm of $4 \times 180/\pi^2 n^2$ for different values of n have been tabulated in Table C.

To illustrate the method of calculation a specimen calculation is shown in Table D.

As some of the coefficients are negative and their logarithms are imaginary, I have used the prefix j to denote $\log(-1)$ must be added. If j occurs twice in the logarithms to be added together the two j's of course cancel; if j occurs in the final result the quantity of which it is the logarithm must have the minus sign prefixed.

10. Mr. Caradog Williams has applied the method to the statistics of Eiffel's experiments tabulated on pages 118, 119, of the English translation of his book, and the results of Mr. Williams's calculations are tabulated in Table B.

It will be seen that in some cases the terms after decreasing again increase, so that after the first few terms further convergence does not take place for some time. An examination of the calculations has shown us that these large terms are due to the discontinuity which in the square plate occurs between 35° and 40°. In fact in the expression for P_n given by (12) the value of $\triangle(\triangle y/\triangle x)$ is large in the neighbourhood of the discontinuity and small elsewhere. Consequently the terms contributed by this part of the scale considerably exceed the algebraic sum of the positive and negative ones contributed by the other parts of the scale.

It will be seen that the series derived from these experimental data usually converge more slowly than those proposed from theoretical considerations. On the other hand the results differ so widely in different cases that it hardly appears likely that they could adequately be represented by any better method than the Fourier expansion. In order to estimate the number of terms required to give a fair approximation to the correct result Mr. Caradog Williams has drawn up Figs. 1, 2 and 3, in which the original graph of Eiffel's tabulated results is shown side by side with the curves obtained by taking a limited number of terms of the Fourier expansion. It will be seen that in the Fourier curves the fluctuations are in general rather less marked than in the original curves, the former curves lying as a rule on the concave side of the latter. This difference is no doubt in some measure due to the substitution of a polygonal outline for a continuous curve in the evaluation of the coefficients. This polygonal figure lies on the concave side of the actual curve and when the higher terms of the expansion are neglected, we naturally expect to get a curve lying within the original one.

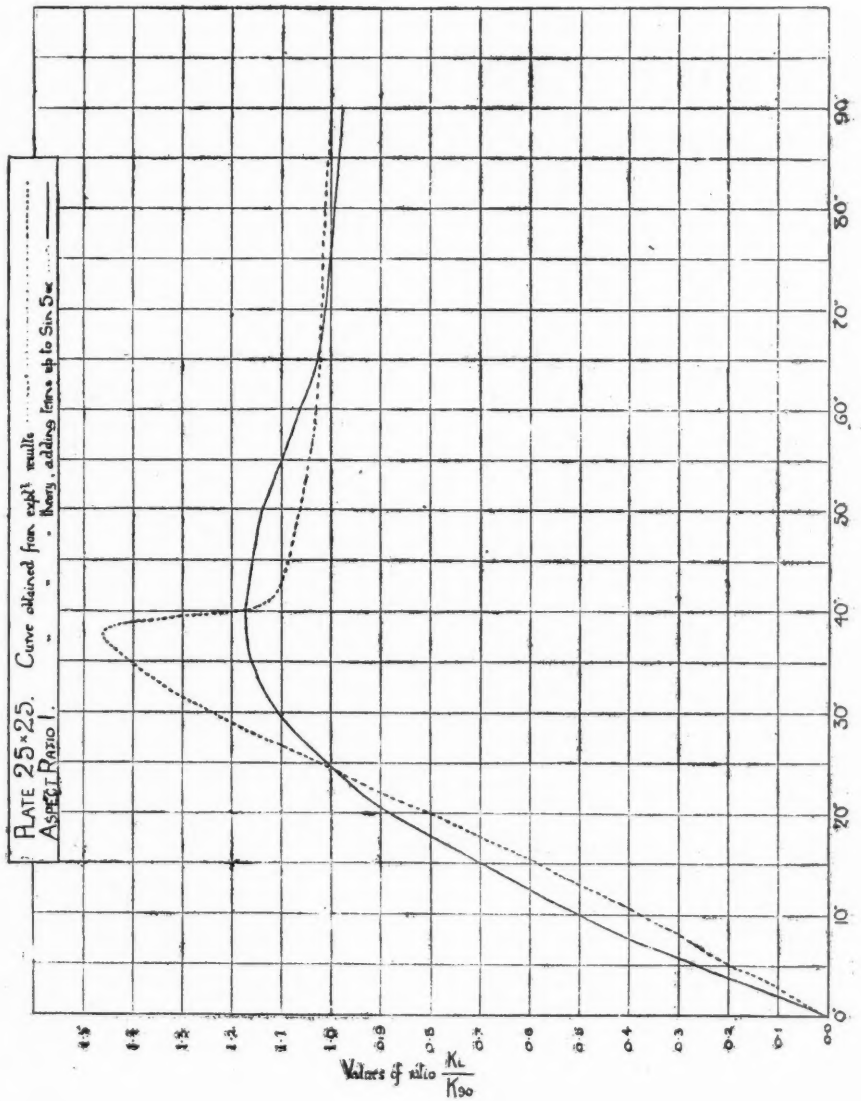
PLATE 25×25. Curve obtained from expt^l results
ASPECT RATIO 1.
Theory, adding lines up to Sin 5α
Values of ratio $\frac{K_L}{K_{90}}$
90°
80°
70°
60°
50°
40°
30°
20°
10°
0°

FIG. 1.

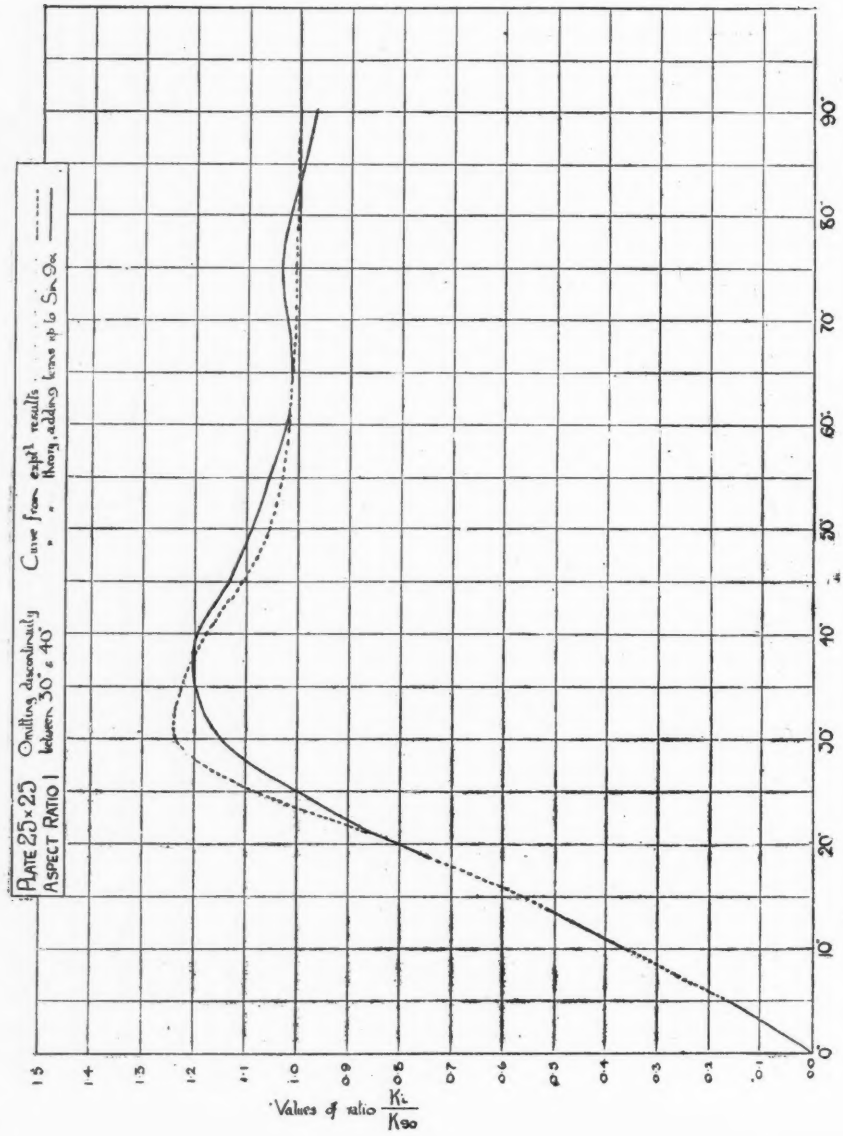
PLATE 25×25
ASPECT RATIO 1
Omitting discordantly
between 30° & 40°
Curve from expl. results
" " theory, added to suit 90°
Values of ratio $\frac{K_i}{K_{90}}$

FIG. 2.

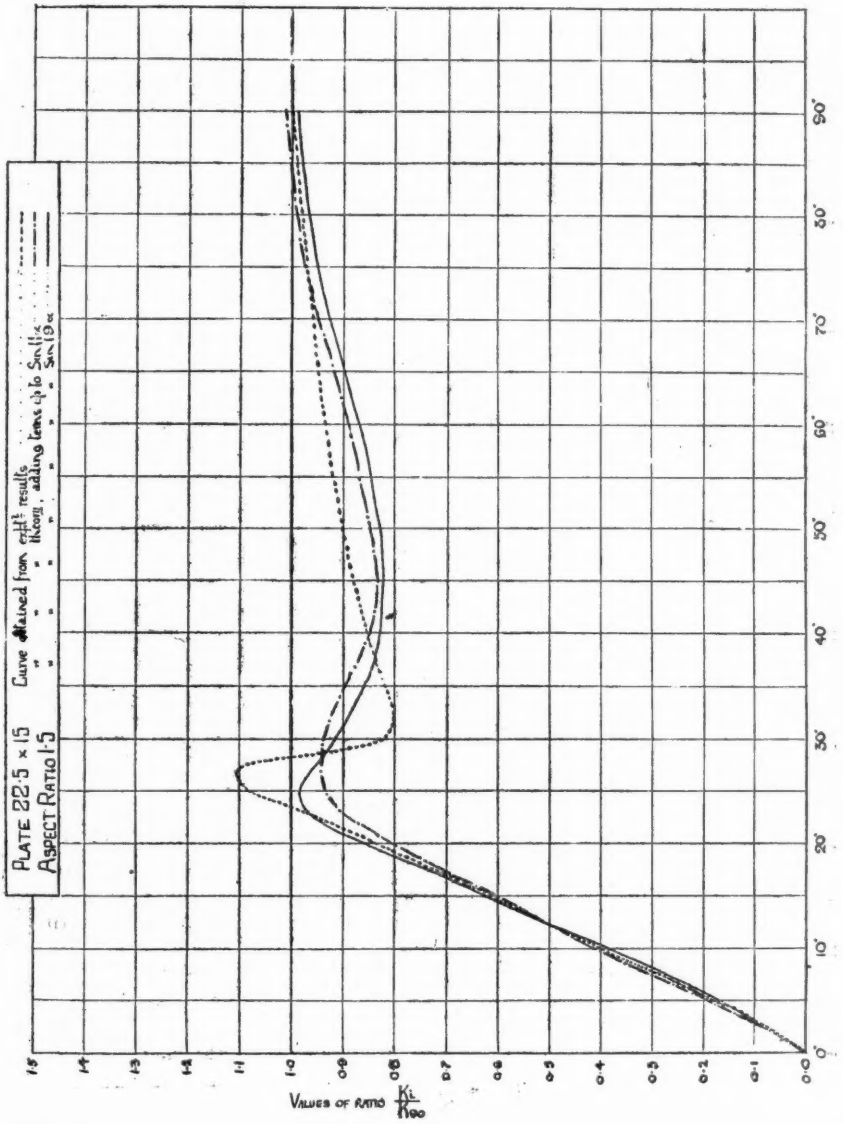
PLATE 22·5 × 15
ASPECT RATIO 1·5
Curve derived from eqpt results
" " " " Eiffel, adding terms up to Sin 11α
" " " " " " Sin 13α
90°
80°
70°
60°
50°
40°
30°
20°
10°
0°
1·5
1·4
1·3
1·2
1·1
1·0
0·9
0·8
0·7
0·6
0·5
0·4
0·3
0·2
0·1
0·0
VALUES OF RATIO Ki/Kα

FIG. 3.

It is obvious that other methods of evaluating the coefficients are possible in which this defect is not liable to occur, but as previously stated, the calculations in such cases would probably be more difficult and it would hardly be worth while to make the attempt at present.

On the whole I should think that in most applications it would be of little use to carry the calculations beyond two places of decimals and *that a formula of the form* $P_1 \sin \alpha + P_3 \sin 3\alpha + P_5 \sin 5\alpha$ *will be sufficient for most practical purposes* except in the case of the square plate and a few other instances, in which it may be necessary for greater accuracy to retain $\sin 7\alpha$, $\sin 9\alpha$, or even further terms.

It will be seen that all the formulæ expanded in Table A are represented with sufficient accuracy by the first three terms, the neglected coefficients in no case exceeding about 2 per cent. of the coefficient of $\sin \alpha$.

11. A possible criticism of this conclusion may be anticipated. It will, no doubt, be said that when all is said and done we have only got a result which can be reduced to the form $A \sin \alpha + B \sin^3 \alpha + C \sin^5 \alpha$. This conclusion, however, applies to nearly every practical application of Fourier's series and still more so to other expansions used by mathematical physicists, in particular, to the use of spherical harmonies. It is certain that if equally good results could be obtained more easily by other methods we should do well to make a bonfire of most of the existing literature of the Fourier analysis. The important point is that the best results are obtained when the Fourier mill is used to grind out the successive terms of the expansion; when this has been done, it is immaterial in what form they are exhibited. The justification for this statement lies in the fact that Fourier's method gives identically the same results that would be obtainable by the method of least squares.

12. *The expressions for the lift and drag,* expressed as fractions of $P(90°)$, reduce to the forms

$$\text{Lift} = \tfrac{1}{2}(P_1 + P_3) \sin 2\alpha + \tfrac{1}{2}(P_3 + P_5) \sin 4\alpha + \tfrac{1}{2}(P_5 + P_7) \sin 6\alpha + \ldots$$
$$\text{Drag} = \tfrac{1}{2}P_1 + \tfrac{1}{2}(P_3 - P_1) \cos 2\alpha + \tfrac{1}{2}(P_5 - P_3) \cos 4\alpha + \ldots$$
$$\ldots \qquad (13)$$

The lift and drag coefficients are calculated in the case of the square plate, the rectangle of aspect ratio 9, and rectangle of aspect ratio 1/3, and the results are tabulated in Table F.

13. *The constant term* Q_0 *of the cosine series* for $f(\alpha)/\sin \alpha$ does not lend itself to calculation by this method, for if we assume $f(\alpha)$ to be a linear function of α between two consecutive plotted points on its graph, we obtain an integral which cannot be evaluated in finite form. The only way, therefore, is to assume a polygonal outline for the graph of $f(\alpha)/\sin \alpha$ in which case Q_0 is given by the trapezoidal rule. As, however, this is different to assuming a polygonal outline for the graph of $f(\alpha)$ it is obvious that the difference might have a considerable effect on the subsequent calculation of the terms of the Q series from those of the P series. Mr. Caradog Williams has therefore calculated the cosine series in the case of the square plate and the rectangle aspect ratio 1.5, and it will be seen that the differences in this case are unimportant. The results are given in Table E. This objection does not of course apply in the case of the theoretical expansions of Table A.

14. *Formulæ for the pressure in oblique positions.*—If a rectangle whose aspect ratio is 6 be rotated in its own plane through 90°, it will be described in the new position as having an aspect ratio of $\frac{1}{6}$. If, however, it be turned through an intermediate angle β, it will be in an oblique position relative to the wind, as exemplified by the case in which an aeroplane is struck by a side wind or is moving sideways or turning about a point in front of or behind its main planes.

From the absence of experimental data in the literature which I have consulted, it would appear that little has been done in investigating the pressures in these oblique positions. But considerations of symmetry show that the proper form of the Fourier series to use in such cases is

$$a_0 + a_2 \cos 2\beta + a_4 \cos 4\beta \ . \ . \ .$$

where β may be called the azimuth of the wind relative to the plane. When the pressure function is expanded in sines of multiples of α *each of the coefficients must be expanded in this form.* The result will be a *double Fourier series* giving the pressure on the plane as a fraction of $P\,(90°)$ for any direction whatever of the relative wind.

For small angles of attack in which the value of $P\,(\alpha)$ is proportional to $\sin \alpha$ the proper form to assume will still be

$$(a_0 + a_2 \cos 2\beta + a_4 \cos 4\beta + \ . \ . \ .) \sin \alpha$$

If it should be possible to neglect the terms involving $\cos 4\beta$ and higher multiples, the expression for any such coefficient will assume the form

$$c_1 \cos^2\beta + c_2 \sin^2\beta \qquad . \quad . \quad . \quad . \qquad (14)$$

where c_1 and c_2 represent the values of the corresponding coefficient in pterygoidal and apheroidal aspects for the same plane. For example, if the dimensions of a rectangular plane are 15cm. and 90cm., we should multiply the coefficients in the fifth row by $\cos^2\beta$ and those in the last row by $\sin^2\beta$ and add, thus giving

$$(1.133 \cos^2\beta + 1.175 \sin^2\beta) \sin \alpha + (0.267 \cos^2\beta + 0.144 \sin^2\beta) \sin 3\alpha + \ . \ . \ .$$

There is, however, no evidence that the coefficients above $\cos 2\beta$ can be neglected, and, on the contrary, such an assumption, when applied to a square lamina, would lead to the result that the pressure was the same, for a given angle of attack, whether the square was placed with a diagonal or with one of its edges perpendicular to the wind. These are questions which can only be determined by experiment. I should expect to find that the form (14) would be fairly accurate if applied to an elliptic lamina, but for a rectangle it would not be surprising to find discontinuities for certain values of the angle β. For a square plate the series must necessarily be taken of the form

$$a_0 + a_4 \sin 4\beta + a_8 \sin 8\beta + \ . \ . \ .$$

Further than this it is impossible to proceed without actual experimental evidence.

15. *Applications of the Method of Least Squares.*—The well known connection between Fourier's Theorem and the method of Least Squares may be used with advantage in the approximations for aerodynamical coefficients. Taking, for example, an expansion

$$f\,(\alpha) = a_1 \sin \alpha + a_3 \sin 3\alpha + \ . \ . \ . \ + a_n \sin n\alpha + \ . \ . \ .$$

the mean square of the error resulting from taking the first n terms of the series as an approximation for $f\,(\alpha)$ in the interval from 0 to $\frac{1}{2}\pi$ is

$$\frac{2}{\pi} \int_0^{\pi/2} (f\,(x) - a_1 \sin x - a_3 \sin 3x \ . \ . \ . \ - a_n \sin nx)^2 \, dx \ . \qquad (15)$$

Differentiating, we see that this is a minimum by the variation of a_n when

$$0 = \int_0^{\pi/2} (f\,(x) - a_1 \sin x - a_3 \sin 3x \ . \ . \ . \ - a_n \sin nx) \sin nx \, dx \qquad (16)$$

This expression represents the ordinary rule for determining the coefficients in

Fourier's series by multiplying by $\sin n\alpha$ and integrating. Moreover, the mean square of the error is equal to

$$\frac{2}{\pi}\int_0^{\pi/2}\left\{f(x)\right\}^2 dx - \tfrac{1}{2}(a_1^2 + a_3^2 + \ldots + a_n^2) \quad . \quad (17)$$
$$= \tfrac{1}{2}(a^2_{n+2} + a^2_{n+4} + a^2_{n+6} + \ldots)$$

We have chosen a series of odd sines as an example, but the same method applies to any series. From this the degree of accuracy of any approximation can be estimated.

It is important to notice that according to the method of least squares, the sum of any number of terms of the series, or even a single term or a collection of terms picked out arbitrarily gives the best possible approximation to the function when the values of the coefficients are those determined by Fourier's Theorem. For example, the value of $P_1 \sin\alpha$ will make the mean square of the error a minimum in the interval from 0 to 90° when it is sought to replace $f(\alpha)$ by a single term containing $\sin\alpha$. It therefore gives the best approximation for the whole range of values from 0 to 90°. It is, therefore, not however the best value to take when only small angles have to be considered.

But the method of least squares can also be used to obtain the series which gives the best approximation to a function $f(\alpha)$ over any interval whatever. The coefficients will still be obtainable by the ordinary method of multiplying and integrating as is evident from equation A. The only difference is that unless the limits of integration are 0 and 90°, or similar angles, the integrated products of sines or cosines will not vanish and the coefficients will therefore have to be found by solving a number of simultaneous equations. Examples of this method will be given later on. The following application may be found useful in aerodynamical calculations.

PROBLEM.—*Given the value of* $f(\alpha)$ *in the form of a Fourier's series between the limits* 0 *and* 90°, *to find the value of* A *in order that the single term* $A \sin\alpha$ *should give the best possible approximation to* $f(\alpha)$ *in a given limited interval, say, between* 0 *and* 30°.

Taking the original given expansion to be $\Sigma P_n \sin n\alpha$ with n odd, and assuming $A = P_1 + K$, the method of least squares gives

$$\int (K\sin^2 x - P_3 \sin x \sin 3x - P_5 \sin x \sin 5x \ldots)\, dx = 0 . \quad (18)$$

For the limits 0 and 30°, which are often required in aeronautical work, I find that

$$K\left(\frac{2\pi\sqrt{3}}{9} - 1\right) = \frac{P_3}{2} + \frac{P_5}{2} + \frac{P_7}{4} - \frac{P_9}{20} - \frac{P_{11}}{5} - \frac{P_{13}}{7} \text{ etc.} \quad . \quad (19)$$

It will probably be better to employ this or some similar formula where it is required to employ the sine rule for small angles of attack rather than to use a series, the convergence of which is doubtful, for the limit of $f(\alpha)/\sin\alpha$ when α approaches zero.

16. *Formulæ for the Centre of Pressure.*—According to "Joessel's Formula" the distance of the centre of pressure of a plane lamina in front of the centre of the area is given by

$$0.3\,(1 - \sin\alpha)$$

referred to a lamina of which the whole breadth is unity. In some books the coefficient is quoted as 0.33 instead of 0.3.

This formula can only hold good if α is in the first quadrant. If α is in the second quadrant it is obvious that the corresponding form must be $-0.3\,(1-\sin\alpha)$.

To obtain a formula which shall hold good in all four quadrants it is easily seen from considerations of symmetry that the expression must be expanded in cosines of odd multiples of α between the limits 0 and $\frac{1}{2}\pi$. We obtain on multiplying and integrating

$$0.3\,(1-\sin\alpha) = 0.3 \times \frac{4}{\pi}\left\{\frac{\cos\alpha}{1.2} + \frac{\cos 3\alpha}{2.3} + \frac{\cos 5\alpha}{5.6} + \frac{\cos 7\alpha}{6.7} + \frac{\cos 9\alpha}{9.10} + \frac{\cos 11\alpha}{10.11} \text{ etc.}\right\} \quad . \quad . \quad . \quad (20)$$

According to the Kirchoff-Rayleigh theory of discontinuous liquid motion the corresponding expression is

$$\tfrac{3}{4}\,\frac{\cos\alpha}{4+\pi\sin\alpha}$$

and the coefficients in the Fourier expansion may be determined by a formula of reduction similar to that described in § 8.

Another formula, sometimes described as Soreau's formula, is

$$\frac{1}{4\,(1+2\tan\alpha)}$$

which may be written

$$\frac{\cos\alpha}{4\,(\cos\alpha+2\sin\alpha)}$$

The integrals required for evaluating the Fourier coefficients may be evaluated by the reduction formula

$$\frac{5\cos n\alpha - 6\cos(n-2)\alpha + 5\cos(n-4)\alpha}{\cos\alpha+2\sin\alpha} = 4\cos(n-2)\alpha\,(\cos\alpha-2\sin\alpha) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

The numerical values of the coefficients in these expressions have been formed by Mr. Creak and are given in Table G.

The corresponding coefficients calculated from several of Eiffel's experimental results for plane areas are given in the same table and have been calculated by Mr. Caradog Williams. The method employed in this case is the same as in the calculation of the pressure coefficients. If $\phi(\alpha)$ is the function to be expanded and we assume

$$y = \phi(\alpha) = C_1\cos\alpha + C_3\cos 3\alpha + C_5\cos 5\alpha \ . \ . \ .$$

we obtain

$$\frac{\pi}{4}C_n = \int_0^{\pi/2}\phi(x)\cos nx\,dx = \left[\frac{\phi(x)\sin nx}{n}\right]_0^{\pi/2} - \int_0^{\pi/2}\frac{\phi^1(x)\sin nx}{n}\,dx \quad . \quad . \quad . \quad . \quad . \quad (22)$$

In this case $\phi(\pi/2) = 0$ so that the terms in square brackets therefore vanish.

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If now in the interval between two observed data we replace $\phi^1(x)$ by its mean value $\triangle y/\triangle x$, we obtain

$$\frac{\pi}{4} C_n = \Sigma \left[\frac{\Delta y}{\Delta x}\right]_r^{r+1} \frac{\cos nx_{r+1} - \cos nx_r}{n^2}$$

$$= -\frac{1}{n^2} \Sigma \cos nx \, \Delta \left(\frac{\Delta y}{\Delta x}\right)$$

In this formula, angles are measured in radians. In practical work Δx is necessarily measured in degrees, so that when thus measured the formula becomes

$$C_n = \frac{4 \times 180}{\pi^2 n^2} \Sigma \cos nx_r \left\{ \left[\frac{\Delta y}{\Delta x}\right]_{r-1}^{r} - \left[\frac{\Delta y}{\Delta x}\right]_r^{r+1} \right\} \qquad (23)$$

A difficulty arises from the fact that when $\alpha = 0$ the pressure vanishes and consequently $\phi(0)$ is indeterminate. Consequently the lowest value of α for which $\phi(\alpha)$ is tabulated in Eiffel's experiments is 2°. This difficulty is, however, not so serious a one as might have been expected. The only indeterminate quantity is the term

$$\left[\frac{\Delta y}{\Delta x}\right]_0^{2^\circ} \frac{\cos n\, 2^\circ - 1}{n^2}$$

Unless n is large, $\cos n\, 2^\circ - 1$ is very small, and so this term is not very important. Mr. Caradog Williams has dealt with this question in two different ways, first, by neglecting this term altogether, and secondly, by assuming that the value of $\triangle y/\triangle x$ between the first two tabulated data remains the same down to $\alpha = 0$. In the former case the curve is assumed to be horizontal from $\alpha = 0$ up to the first tabulated value; in the latter it is replaced by straight line from $\alpha = 0$ up to the second tabulated value.

The difference in the numerical values of the coefficients is however small. For example, in the case of the rectangle 15 × 45cm., aspect ratio $\frac{1}{3}$, we have found that in the former case $C_{11} = .009233$ and that in the second case $C_{11} = .007783$. The expansions converge rather more slowly in the case of the experimental data than in the theoretical formulæ. This is, of course, due to the fact that according to Eiffel's curves $\phi^1(\alpha)$ is considerable when α is small and of course it changes sign with α. The formulæ may therefore be expected to agree fairly well with the assumed tabulated data except when α is small.

17. *Formula for the Moment of Resultant Pressure.*—Assuming Duchemin's formula for the resultant pressure and Joessel's formula for the centre of pressure, the moment of the resultant pressure about the centre of figure is given by

$$\frac{3}{10} (1 - \sin\alpha) \frac{2 \sin\alpha}{1 + \sin^2\alpha} \quad . \quad . \quad . \quad . \quad . \quad (24)$$

taking as a unit the moment which a perpendicular thrust P_{90} would have when acting on an arm = whole breadth of plate.

As in §4 the moment of the resultant pressure can be expanded in an even sine series

$$= \Sigma C_{2n} \sin 2n\alpha$$

where

$$C_{2n} = \frac{4}{\pi} \int_0^{\pi/2} \frac{3}{10} (1 - \sin\alpha) \frac{2 \sin\alpha}{1 + \sin^2\alpha} \sin 2n\alpha \, d\alpha$$

The formula of reduction in this case becomes

$$\frac{\sin n\alpha - 6\sin(n-2)\alpha + \sin(n-4)\alpha}{1+\sin^2\alpha} = -4\sin(n-2)\alpha\ . \qquad (25)$$

The numerical coefficients in the expansion have been calculated by Mr. Creak and are given in Table H.

As in previous examples, however, if approximate values are taken the coefficients calculated from this reduction formula ultimately form a divergent sequence.

The calculation of the corresponding series for other theoretical formulæ, as also for the results of experimental data, presents no special difficulty and it hardly appears worth while to consider any further cases in detail.

18. In a subsequent paper we propose to discuss the best way of applying Fourier's series to the aerodynamical coefficients of cambered planes, aeroplane wings and complete systems of aeroplane surfaces. Such cases present a number of peculiarities which will require careful consideration.

It would be premature to express any opinion as to the practical value of the Fourier method, because everything must necessarily depend on the class of investigations for which their utility may have to be considered. All that the present paper claims to do is to give a general idea of the kind of results to which this method leads, and any special peculiarities presented by the methods most convenient for obtaining them. It will, I think, be seen that the methods here described involve several points of difference from the subject matter contained in existing papers and text books. It may be further stated that most of these differences were not anticipated when the present investigation was commenced, with the result that what was thought to be a short piece of work has grown into a long one.

TABLE A.—PRESSURE COEFFICIENTS (THEORETICAL).

$P\alpha/P_{90}=\Sigma P_n \sin n\alpha$ (n odd).	P_1	P_3	P_5	P_7	P_9	P_{11}
Duchemin's formula ...	1·1716	·2010	·0345	·0059	·0010	·0002
Gerlach's formula (modification of Rayleigh)	1·0794	·0979	·0252	·0097	·0047	·0026
Renard's formula (modified for $P_{90}=1$)	1·2893	·1357	—·0742	·0208	—·0129	·0078
Soreau's formula ...	1·25	·25				
Eiffel's formula (1907) $P\propto\alpha$ up to 30° and $P=1$ from $\alpha=30°$ to $\alpha=90°$	1·2159	·2702	·0486	—·0248	—·0300	—·0100
$P=2\sin\alpha$ up to 30° $P=1$ from $\alpha=30°$ to $\alpha=90°$	1·2180	·2757	·0551	—·0197	—·0276	—·0100

TABLE B.—PRESSURE COEFFICIENTS.
EIFFEL'S EXPERIMENTAL RESULTS FOR PLANE AREAS.

$P\alpha/P_{90} = \Sigma P_n \sin n\alpha$ (n odd).	P_1	P_3	P_5	P_7	P_9	P_{11}
Square Plate 25 × 25cm. Aspect Ratio = 1 ...	1·3163	·4137	·0794	—·0815	—·0830	—·0024
Rectangle 22.5 × 15cm. Aspect Ratio = 1.5 ...	1·1554	·2655	·1385	·0372	—·0208	—·0493
Rectangle 30 × 15cm. Aspect Ratio = 2 ..	1·1052	·1299	·1342	·0703	·0313	—·0096
Rectangle 45 × 15cm. Aspect Ratio = 3 ...	1·0059	·2193	·1157	·0572	·0335	·0058
Rectangle 90 × 15cm. Aspect Ratio = 6 ...	1·1327	·2666	·1264	·0741	·0478	·0254
Rectangle 90 × 10cm. Aspect Ratio = 9 ...	1·1462	·2399	·1319	·0790	·0518	·0309
Rectangle 15 × 45cm. Aspect Ratio = $\frac{1}{3}$ ...	1·2661	·2566	—·0733	—·0893	—·0080	·0214
Rectangle 15 × 90cm. Aspect Ratio = $\frac{1}{6}$ ...	1·1749	·1439	—·0878	—·0361		

TABLE C.

n	1	3	5	7	9	11	13	15	17	19	21
$\log \frac{4 \times 180}{\pi^2 n^2}$	1·8631	·9089	·4652	·1729	1·9546	1·7803	$\bar{1}$·6352	$\bar{1}$·5109	$\bar{1}$·4022	$\bar{1}$·3056	$\bar{1}$·2187

TABLE D.—SPECIMEN CALCULATION.
SQUARE PLATE 25 × 25cm., ASPECT RATIO 1.

α	y	$y_n - y_{n-1}$	$x_n - x_{n-1}$	Quotient.	Factor F.	log F.
0	0				—·033	
		·165	5	·033		
5	0·165				—·006	j $\bar{3}$·7782
		·195	5	·039		
10	0·36				—·0055	j $\bar{3}$·7404
		·445	10	·0445		
20	0·805				·0010	$\bar{3}$·0000
		·435	10	·0435		
30	1·24				·0115	$\bar{2}$·0607
		·16	5	·032		
35	1·40				·012	$\bar{2}$·0792
		·06	3	·02		
38	1·46				·165	$\bar{1}$·2175
		—·29	2	—·145		
40	1·17				—·127	j $\bar{1}$·1038
		—·09	5	—·018		
45	1·08				—·0147	j $\bar{2}$·1673
		—·05	15	—·0033		
60	1·03				—·0023	j $\bar{3}$·3617
		—·015	15	—·001		
75	1·015				·000	
		—·015	15	—·001		
90	1·00				—·001	j $\bar{3}$·0000

CALCULATION OF P_3.

α	5	10	20	30	35	38	40	45	60	90
3α	15	30	60	90	105	114	120	135	180	270
Nearest acute					75	66	60	45	0	j 90
log C_3	·9089	·9089	·9089	·9089	·9089	·9089	·9089	·9089	·9089	·9089
log sin 3α	$\bar{1}$·4130	$\bar{1}$·6990	$\bar{1}$·9375	0·0000	$\bar{1}$·9849	$\bar{1}$·9607	$\bar{1}$·9375	$\bar{1}$·8495	omit (logo)	j 0·0000
log F	j $\bar{3}$·7782	j $\bar{3}$·7404	$\bar{3}$·0000	$\bar{2}$·0607	$\bar{2}$·0792	$\bar{1}$·2175	j $\bar{1}$·1038	j $\bar{2}$·1673	j $\bar{3}$·3617	j $\bar{3}$·0000
log X	j $\bar{2}$·1001	j $\bar{2}$·3483	$\bar{3}$·8464	$\bar{2}$·9696	$\bar{2}$·9730	·0871	j $\bar{1}$·9502	j $\bar{1}$·9257		$\bar{3}$·908 9
	−0·01259	−0·0223	0·007021	0·09324	0·09397	1·222	−0·8915	−0.08428		0·008108

Positive
·007021
·093240
·093970
1·222000
·008108
1·424339
— 1·010670
·413669 = P_3

Negative
— ·01259
— ·02230
— ·89150
— ·08428
—1·01067

TABLE E.—PRESSURE COEFFICIENTS (COSINE SERIES).

$P\alpha/P_{90} = \sin\alpha \Sigma Q_{2n} \cos 2n\alpha$	Q_0	Q_2	Q_4	Q_6	Q_8	Q_{10}
Square Plate 25 × 25cm. Aspect Ratio = 1 ...	1·6703	·6883	—·1354	—·2245	—·0833	·0464
Rectangle 22.5 × 15cm. Aspect Ratio = 1.5 ...	1·5397	·7245	·2016	—·0682	—·1450	—·0986

Values of P_{n+1} obtained from above. (Compare Table B.)

P_{n+1} = coefficient of $\sin(n+1)\alpha$.	P_1	P_3	P_5	P_7	P_9	P_{11}
Square Plate 25 × 25cm. Aspect Ratio = 1 ...	1·3361	·4119	·0445	—·0706	—·0649	—·0086
Rectangle 22.5 × 15cm. Aspect Ratio = 1.5 ...	1·1774	·2615	·1349	·0384	—·0232	—·0440

TABLE F.—LIFT COEFFICIENTS.

Lift$/P_{90} = \Sigma L_{2n} \sin 2n\alpha$.	L_2	L_4	L_6	L_8	L_{10}	L_{12}
Square Plate 25 × 25cm. Aspect Ratio = 1 ...	·8650	·2465	—·0011	—·0822	—·0427	
Rectangle 90 × 10cm. Aspect Ratio = 9 ...	·6930	·1859	·1054	·0654	·0413	.0279
Rectangle 15 × 45cm. Aspect Ratio = ⅓ ...	·7604	·0907	—·0813	—·0487	·0067	·0101

DRAG COEFFICIENTS.

Drag$/P_{90} = \Sigma D_{2n} \cos 2n\alpha$.	D_0	D_2	D_4	D_6	D_8	D_{10}
Square Plate 25 × 25cm. Aspect Ratio = 1 ...	·6582	—·4513	—·1671	—·0805	—·0007	·0403
Rectangle 90 × 10cm. Aspect Ratio = 9 ...	·5731	—·4531	—·0540	—·0264	—·0136	—·0104
Rectangle 15 × 45cm. Aspect Ratio = ⅓ ...	·6330	—·5057	—·0907	—·0180	·0407	·0147

TABLE G.—CENTRE OF PRESSURE COEFFICIENTS.

(Distance of centre of pressure from centre of figure = full breadth of plate $\times \Sigma C_n \cos n\alpha$ (n odd).

	C_1	C_3	C_5	C_7	C_9	C_{11}
Joessel's formula ...	·1910	·0637	·0127	·0091	·0042	·0035
Rayleigh's formula ...	·1439	·0200	·0075	·0038	·0023	.0014
Soreau's formula ...	·1555	·0311	·0176	·0094	·0066	·0044

EIFFEL'S EXPERIMENTAL RESULTS FOR PLANE AREAS.

Square Plate 25 × 25cm. Aspect Ratio = 1 ..	·1790	·0553	·0401	·0145	·0097	·0052
Rectangle 45 × 15cm. Aspect Ratio = 3 ..	·1383	·0301	·0284	·0225	·0152	·0093
Rectangle 90 × 15cm. Aspect Ratio = 6 ...	·1388	·0261	·0235	·0173	·0126	·0089
Rectangle 15 × 45cm. Aspect Ratio = $\frac{1}{3}$ ...	·2054	·0281	·0010	·0110	·0240	·0078
Rectangle 15 × 90cm. Aspect Ratio = $\frac{1}{6}$ ...	·1926	—·0043	—·0055	·0131	·0082	—·0035

TABLE H.—MOMENT OF RESULTANT PRESSURE COEFFICIENTS.

(Moment of resultant pressure about centre of figure = $\Sigma G_{2n} \sin 2n\alpha$ taking as unit the moment of the normal pressure P_{90} acting on an arm equal to the full breadth of the lamina.)

	G_2	G_4	G_6	G_8	G_{10}	G_{12}	G_{14}
From Duchemin's and Joessel's formulæ ...	·0935	·0515	·0119	·0055	·0016	·0013	·0006
From Soreau's formulæ	·0950	·0267	·0082	·0034	·0021	·0011	·0009

LONG TAILS VERSUS SHORT TAILS IN AERODYNAMICS.

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1. Consider any form of aircraft provided with a small rudder plane placed behind its centre of gravity and at a certain distance l from it, which we may speak of as the length of the tail. When this tail is subjected to a given wind pressure, the moment of this pressure about the centre of gravity, tending to turn the machine round, is proportional to l. From this point of view a long tail is preferable to a short one.

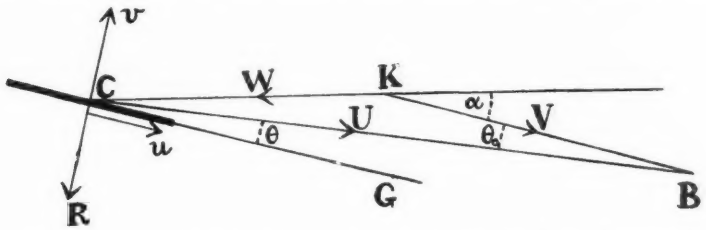
On the other hand, the pressure of the air on the tail plane depends on the normal velocity of the wind relative to it. Hence the normal velocity of the plane itself is limited in magnitude, and the angular velocity with which the machine turns about its centre of gravity must therefore be less when l is large than when l is small; that is, the long-tailed machine turns more slowly than the one with a short tail, and this is a disadvantage.

We thus have two conditions which are to a certain extent incompatible, and it becomes interesting and useful to work out a problem in which the joint effect of these conditions is taken into account. In order to avoid complications it appeared to be desirable in the first instance to consider the simplest possible form of system furnished with a tail. The system discussed in the following paper might perhaps be described as a "dart," and may be roughly illustrated by an ordinary arrow. It is assumed to be a heavy body experiencing no resistance itself from the air, and acted on by no forces, but furnished with a rudder plane, the air pressure on which follows the sine law of resistance when the plane is moving in a direction other than tangential relative to the air. In this case the pressure on the tail is wholly employed in overcoming the inertia of the system and changing its motion so as ultimately to bring the direction of motion and the line joining the plane to the centre of gravity into coincidence; or, in ordinary language, to make the system fly in the direction in which it is pointing. The problem presented by an aeroplane or an airship is of course much more complicated, because the pressure on the rudder is largely employed in overcoming the air resistances on the machine itself. In the case of an aeroplane which cannot turn curves without banking, the discussion of the action of a vertical rudder brings us back to the problem of lateral oscillations and lateral stability, while the effect of a horizontal rudder is considered in connection with longitudinal stability.

2. A plane lamina, centre C, so narrow that for moderate angles of attack we may use the sine law for the air pressure and ignore the shift of the centre of pressure, is attached to a system so that the whole mass is M, and the centre of mass is at G, outside the lamina but in its plane. Originally the system is moving with a given velocity V along the lamina and with no rotation. A wind of constant velocity W is suddenly generated at a small angle α with the direction of motion, backwards. To find the oscillation set up, and the best position for the tail, whose size is given. The motion is supposed two dimensional—in the plane of the paper—and there are no applied forces.

It will simplify the work if we consider, not the *actual* motion of the system, but the motion *relative to the air*. The initial relative velocity can be found by

the following construction (see Fig.). Measure off CK containing W units of length, so that KC represents W, the velocity of the wind; and draw KB parallel to the initial direction of CG containing V units of length, and therefore representing the initial velocity V of the dart. Then CB represents the initial velocity U of the dart relative to the air, and similarly BC, which is equal and opposite to CB,* represents the relative velocity with which the air appears to blow on



the lamina initially. The problem of the relative motion thus becomes the same as that of a dart originally projected in the direction CB with a velocity U measured by the length CB. If u v are the relative velocity components of the dart along and perpendicular to the lamina, θ the inclination of the lamina to the fixed direction CB, θ_0 the initial value of θ so that θ_0 = angle KBC, then we have initially

$$u_0 = U \cos \theta_0 = W \cos \alpha + V,$$
$$v_0 = U \sin \theta_0 = W \sin \alpha;$$

and if α is small, we may take

$$u_0 = W + V; \ \theta_0 = \frac{W}{W + V} \alpha; \ v_0 = W\alpha \ . \quad . \quad . \quad . \quad (1)$$

The quantities u, v, θ, define the motion of the dart relative to the air. It is clear, however, that θ represents the *actual* direction of CG at any moment.

3. If CG, the distance of the centre of the lamina *behind* the centre of mass of the system is l, and Mk^2 is the moment of inertia of the whole system about an axis through G normal to the plane of the motion, the equations of motion are as follows:—

$$\left.\begin{aligned} \frac{du}{dt} + v\frac{d\theta}{dt} - l\left(\frac{d\theta}{dt}\right)^2 &= 0; \\ \frac{dv}{dt} - u\frac{d\theta}{dt} - l\frac{d^2\theta}{dt^2} &= -\frac{R}{M}; \\ Mk^2\frac{d^2\theta}{dt^2} &= -lR; \end{aligned}\right\} \quad . \quad . \quad . \quad (2)$$

where R is the air resistance. As the lamina is narrow, we assume the sine law of resistance and put

$$\frac{R}{M} = Av\,(u^2 + v^2)^{1/2}, \quad . \quad . \quad . \quad . \quad (3)$$

* NOTE.—If the velocities of two moving bodies be represented in magnitude and direction by two sides of a triangle, both drawn *from* their common vertex, then the third side represents their relative velocity. Thus in the above figure where KC and KB represent the velocities of the wind and dart, the relative velocity is represented by CB or BC according to whether we are considering the relative motion of the dart or of the wind. This simple construction for relative velocity can be easily verified by supposing two bodies to start simultaneously from the point K with the velocities W and V; then in a unit of time they will have arrived at C and B, so that the line CB will show the change of relative position in unit time, and will therefore represent the relative velocity. This explanation appears necessary because the treatment of relative velocity in many text books is very confusing, and it is our experience that even *advanced students* frequently make mistakes in consequence.

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where A is the reciprocal of a length depending upon the size, shape and aspect of the lamina.

We suppose α small, so that θ remains small for a finite oscillation, and v is also small. We therefore neglect squares and products of $d\theta/dt$ and v, and the equations (2) become:—

$$\left.\begin{aligned} \frac{du}{dt} &= 0; \\ \frac{dv}{dt} - u\frac{d\theta}{dt} - l\frac{d^2\theta}{dt^2} &= -Auv; \\ \frac{d^2\theta}{dt^2} &= -\frac{Al}{k^2}uv \end{aligned}\right\} \quad . \quad . \quad . \quad (4)$$

This means that u is constant to the first order of small quantities and equal to U. Eliminating v we get

$$\frac{d^3\theta}{dt^3} + \frac{AU(l^2+k^2)}{k^2}\frac{d^2\theta}{dt^2} + \frac{AlU^2}{k^2}\frac{d\theta}{dt} = 0 \quad . \quad . \quad . \quad (5)$$

The solution of (5) is

$$\theta = P + Q_1 e^{-\lambda_1 t} + Q_2 e^{-\lambda_2 t} \quad . \quad . \quad . \quad . \quad (6)$$

where λ_1, λ_2, are the solutions of the algebraic equation

$$\lambda^2 - \frac{AU(l^2+k^2)}{k^2}\lambda + \frac{AlU^2}{k^2} = 0, \quad . \quad . \quad . \quad . \quad . \quad (7)$$

and P, Q_1, Q_2, are constants defined by the initial conditions.

Measuring the time from the beginning of the disturbance caused by the wind W, we have when $t = 0$,

$$\theta = \theta_0, \frac{d\theta}{dt} = 0, \; v = U\sin\theta_0 = U\theta_0,$$

so that at $t = 0$,

$$\frac{d^2\theta}{dt^2} = -\frac{AlU^2}{k^2}\theta_0, \quad . \quad . \quad . \quad . \quad . \quad (8)$$

to our degree of approximation. After some reduction we find

$$\theta = \frac{\lambda_2 e^{-\lambda_1 t} - \lambda_1 e^{-\lambda_2 t}}{\lambda_2 - \lambda_1}\theta_0 \quad . \quad . \quad . \quad . \quad (9)$$

It follows from (7) that for a stable oscillation we must have l positive, and when this condition is satisfied, *i.e.*, when the lamina is *behind* the centre of mass of the system, the motion is always a stable one, the system gradually attaining some different steady motion. The ultimate direction of relative motion is along CB. The effect of the tail in this case is to alter the orientation of the dart without affecting its relative velocity. The ultimate orientation is also along CB, *i.e.*, along the direction of the initial relative velocity.

4. The solutions of (7) are

$$\lambda_1, \lambda_2 = \frac{AU(l^2+k^2)}{2k^2} \pm \left\{\frac{A^2U^2(l^2+k^2)^2}{4k^4} - \frac{AlU^2}{k^2}\right\}^{1/2} \quad . \quad . \quad (10)$$

The advantage of a very small value of l is that no actual oscillation arises, but a gradual change of direction. But in such a case the decrement of the

deviation from steady motion would be slow. In practice λ_1, λ_2, would be complex, so that a damped oscillation ensues. If we write

$$\left.\begin{aligned} &\lambda_1, \lambda_2 = p \pm iq, \\ \text{where} \quad &p = \frac{AU\,(l^2 + k^2)}{2k^2} \\ \text{and} \quad &q = \left\{ \frac{AlU^2}{k^2} - \frac{A^2U^2\,(l^2 + k^2)^2}{4k^4} \right\}^{1/2} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (11)$$

we obtain for (9)

$$\theta = \theta_0 \frac{(p^2 + q^2)^{1/2}}{q} e^{-pt} \sin\left(qt + \tan^{-1}\frac{q}{p} \right) \quad . \quad . \quad (12)$$

Thus at any moment the amplitude of the oscillation is proportional to

$$\frac{(p^2 + q^2)^{1/2}}{q} e^{-pt} \quad . \quad . \quad . \quad . \quad . \quad (13)$$

In order to get the quickest decrease in the amplitude, no attention being paid to the initial extent of the oscillation, we must make p as large as possible, *i.e.*, by (11), we must make k^2/l^2 as large as possible. If, on the other hand, the important thing is to avoid violent oscillations, then we must make $(p^2 + q^2)^{1/2}/q$ as small as possible. This condition easily reduces to $(l^2 + k^2)^2/k^2l$ as small as possible.

5. In an ideal case, where the tail and the connecting rods are supposed to have no mass, k^2 is independent of l. We then get that in order that the oscillation may die down soon, l must be as large as possible. But in order to avoid violent oscillations, we easily find that l must be as nearly as possible equal to $k/\sqrt{3}$. For practical purposes we conclude that *l must be at least equal to $k/\sqrt{3}$* since diminution below this limit involves a double disadvantage.

In an actual flying system, we may expect that

$$k^2 = a + bl^2 \quad . \quad . \quad . \quad . \quad . \quad (14)$$

where a and b are known positive quantities. We again get that to make the oscillations die down soon, we must make l as large as possible. In order to prevent violent oscillation, we must have

$$\frac{(l^2 + a + bl^2)^2}{(a + bl^2)\,l} \quad . \quad . \quad . \quad . \quad . \quad (15)$$

a minimum. This expression is stationary for values of l given by

$$b\,(1 + b)\left(\frac{l^2}{a}\right)^2 + 3\,\frac{l^2}{a} - 1 = 0,$$

giving the unique possible value

$$\frac{l^2}{a} = \frac{-3 + (9 + 4b + 4b^2)^{1/2}}{2b\,(1 + b)}.$$

Thus the oscillation is least violent for

$$l = \left[a \left\{ \frac{-3 + (9 + 4b + 4b^2)^{1/2}}{2b\,(1 + b)} \right\} \right]^{1/2} \quad . \quad . \quad (16)$$

since this makes (15) a minimum. In practice this value of l would be a lower limit.

When b/a is small, (16) becomes approximately

$$l = \frac{\sqrt{a}}{\sqrt{3}} = \frac{k}{\sqrt{3}} \quad . \quad . \quad . \quad . \quad . \quad (17)$$

agreeing with the result previously found.

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6. Since θ ultimately vanishes it will be seen that the direction of motion ultimately becomes parallel to CB, and this therefore represents the direction of the asymptote of the path. It is, however, important to determine the distance of this asymptote from C, as this represents the total displacement of the dart in a direction perpendicular to CB relative to the air. This displacement is obviously equal to

$$\int (u \sin \theta - v \cos \theta)\, dt \quad . \quad . \quad . \quad . \quad (18)$$

which for our purposes of approximation may be taken to be

$$\int (U\theta - v)\, dt$$

Now from equations (4) we obtain

$$\frac{dv}{dt} - U \frac{d\theta}{dt} = \frac{k^2 + l^2}{l} \frac{d^2\theta}{dt^2} . \quad . \quad . \quad . \quad . \quad (19)$$

This is the equation obtainable by taking moments about the centre of the tail. It follows from this that the total displacement is equal to

$$- \frac{k^2 + l^2}{l} \theta + C_1 + C_2 t \quad . \quad . \quad . \quad . \quad (20)$$

where C_1 and C_2 are constants. But since the initial velocity perpendicular to CB is zero and θ is initially equal to θ_o, the total displacement becomes equal to

$$\frac{k^2 + l^2}{l} \theta_o \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

This is a minimum when $l^2 = k^2$. If, however, $k^2 = a + bl^2$, the condition for a minimum becomes $a = bl^2 + l^2$ and therefore $k^2 = (1 + 2b)\, l^2$. In any case there is a certain length of tail for which the displacement is least.

If, however, we consider the total displacement of the *centre of gravity* we must subtract $l\theta$ from the above amount, and the total displacement sideways now becomes equal to $\theta_o k^2 / l$, showing that a long tail is the best.

It is easy to obtain a generalisation of this last result for the case of any kind of aircraft whose motion may be treated as two dimensional. In such a case there is at every instant a point on the axis of the system through which the resultant air resistance passes. This point takes the place of C the centre of the tail. The main difference is that since the components of air resistance depend both on the linear and angular velocities of the system, and these are not in general proportional throughout the motion (for v is initially finite and $d\theta/dt$ initially zero), the distance of this point from G, which we have hitherto denoted by l will now be variable. The total displacement of G at right angles to the direction of final motion will now be

$$= - \iint \frac{k^2}{l} \frac{d^2\theta}{dt^2}\, dt^2. \quad . \quad . \quad . \quad . \quad (22)$$

We may write this equal to $k^2\theta_o/[l]$ where $[l]$ denotes some quantity such that $1/[l]$ is intermediate between the greatest and least values of $1/l$. In any case it is advantageous from this point of view that $[l]$ should be large.

7. The preceding investigation of the small oscillations is, however, only a particular case of the more general problem of aeroplane stability. A more useful estimate of the initial action of the tail plane can be obtained by expanding θ in powers of the time by means of the exponential series. We thus obtain from (9)

$$\frac{\theta}{\theta_o} = 1 - \frac{\lambda_1\lambda_2 t^2}{2!} + \frac{\lambda_1\lambda_2 (\lambda_1 + \lambda_2)\, t^3}{3!} - \frac{\lambda_1\lambda_2 (\lambda_1^2 + \lambda_1\lambda_2 + \lambda_2^2)\, t^4}{4!} + \quad . \quad . \quad (23)$$

Making use of the formulæ for the sum and product of the roots of the quadratic (7) this gives

$$\theta_0 - \theta = \frac{AlU^2}{k^2}\,\theta_0 \left\{ \frac{t^2}{2} - \frac{AU\,(l^2 + k^2)}{k^2}\,\frac{t^3}{6} + \frac{AU^2\left\{A\,(l^2 + k^2)^2 - lk^2\right\}}{k^4}\,\frac{t^4}{24} - \ldots \right\} . \qquad (24)$$

This equation shows that the initial angular acceleration of the system is greatest when l is large, but in this case the angular acceleration decreases more rapidly than it would do if l were small, thus confirming the general statements at the commencement of the present paper.

ON THE FLIGHT OF LOCUSTS.

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Some time ago when leaving my laboratory I happened to notice a large number of cheels gliding overhead that seemed to me to show something unusual in their mode of flight. I guessed that they were after locusts, a flight of these insects having passed over Agra three days previously. On reaching my house I found, with the help of my binocular, that my conjecture was correct. The following are the notes that I took at the time:—

11th August, 1915, 3.0 p.m. A few small thin clouds. Air rather hazy at a height. Wind S, occasionally moving small branches.

A large number of cheels were gliding in a way that seemed to me unusual. They were at heights between 200 and 400 metres. They were generally slow-flex-gliding and did so in all directions and with frequently changing course. Two were seen rapidly losing height, feet foremost. Occasionally they dived or at least glided steeply downwards at speed. I guessed they were hunting locusts and with my binocular soon saw that several of the cheels kept bending down their heads to eat locusts that they held in their claws.

3.6. A locust noticed in flight. It was about 100 metres overhead. Flapping was with all four wings. It was flapping to the west, but was being carried almost due north by the wind.

3.7. Three locusts seen in flapping flight.

3.10. A locust seen flapping at a height of about 100 metres. It stopped flapping and began gliding with loss of height. Its wings were dihedrally up (The loss of height was very well marked. It may have been at the rate of 1 in 5.)

3.11. A cheel seen gliding steeply downwards for a distance of about 50 metres. Then it caught a locust in its claws and at once changed to slow speed horizontal flight. Making a measurement with my telemeter I found it to be at a height of 300 metres. I could see the yellow colour of the locust while held in the claws of the cheel.

Besides the above noted case I saw two other locusts gliding with loss of height. In one of these cases the wings were clearly seen to be dihedrally up. Otherwise the locusts were, as usual, in continued flapping flight. On the other hand, the cheels, whether soaring with gain or loss of speed or with gain or loss of height, were always gliding in their efforts to catch the locusts. They aimed directly at the insect and caught it as usual in their claws. Occasionally a slight irregularity of flight was noticed just before the cheel caught the locust, but in such cases the cheel was seen to make a sudden change of course as if it had only just seen its prey.

The locusts were not in a thick flight. I never noticed more than two in sight at once. The cluster of cheels kept moving their position, evidently being able to see the locusts from several hundred metres distance.

A few days later I made a similar observation. It was as follows:—

16th September, 1915, 8.45 a.m. In Hewett Park, Agra. A few locusts about. Cheels were catching them at heights between 40 and 150 metres. They did so while in continued gliding flight. On two occasions I saw the cheel make a sudden rotation up round its transverse axis at the moment of catching the locust and occasionally they showed a single flap,

and once three flaps, after catching the insect, as if to regain balance or possibly speed. In gliding down from a height to the insect the cheels always travelled in an apparently straight line. One showed a slight lateral wobble at the moment of seizing its prey. Thus, their flight, with the above exceptions, was entirely gliding whether they were gaining or losing height.

Soaring and Non-Soaring Flight at the Same Time and Place.

Obviously the interesting point about the above observations is that the soaring bird and the non-soaring locust were flying together in the same air. The feebly flying locusts were all being drifted to the north despite their industrious flapping in a westerly direction. In every case their course appeared to be a straight line. At the height above the earth at which they were flying in a light wind there were no local currents capable of affecting their course to an appreciable degree. Nevertheless, the apparently homogeneous air in which they flew was well suited to the soaring flight of the cheels. In the same air one flying creature could only maintain itself by constant flapping, and at once lost height if it glided, while the other maintained its speed, which was a greater speed, by simple gliding whether this gliding was accompanied with gain or loss of height. In a similarly light wind at tree-top level there are probably ascending currents and other air irregularities capable of affecting flight. At this level cheels, when catching flying white ants, always use a great deal of flapping in their flight.

As it does not often happen that soaring and non-soaring fliers are seen in flight side by side the following instances are worth quoting:—

20th August, 1915, at 11.30. Sunshine.

A group of cheels and one eagle were seen circling at a height of about 150 metres. Two crows were present and were teasing the eagle. They were in continued flapping flight. The eagle responded chiefly by a sort of slow double dip movement of its wings. I watched this for several minutes. The wind was west and very light as while I was observing the group of birds did not drift very far away from overhead. At length the crows dived, with wings flexed and retracted, coming down in a swerving course. The crows had been flapping till the moment they dived while the eagle and the cheels were in continued gliding flight. While this was going on two soaring birds of unknown species joined the group. Their wings appeared snow white except that the anterior margins appeared greyish-yellow. A vulture passed flex-gliding just over them at about 200 metres height with the white part of its wings appearing bright yellow.

In this case the crows were above the earth at a height that was very unusual for them. This was not the case in the following observation:—

19th July, 1915. Bombay. 8.0 a.m. Slight sunshine. Three crows seen in flapping flight and one cheel slow-flex-gliding. They were all going in the same direction and at a height of about 10 metres above the house tops. The wind was light and on their beam. Thus, the same air (the cheel was closely following the crows) supported one bird in soaring flight while the other birds had to flap.

In this latter case the non-soaring birds travelled at the same rate as the soaring bird. In the former case they appeared to travel faster. The converse relation may be observed. For instance, I have seen a crow teasing a cheel at the time of commencement of morning soarability. The cheel was circling with an occasional flap. After a few minutes as the air became more soarable the cheel circled without flapping and then glided away at a speed with which the flapping crow was unable to keep up. Some time after the air has become soarable for cheels it becomes soarable for birds of heavier loading such as vultures.

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While this is happening vultures are liable to be teased by cheels. The following is an instance:—

> 11th October, 1913. Calcutta. At 9.55. Air thick. Isolated cumulus. Sunshine. Wind north, slightly moving leaves. A black vulture at 300 metres height being teased by a cheel. The vulture appeared to have a slightly greater speed. It was circling (without flapping) most of the time. The cheel kept approaching by cutting off corners and describing smaller circles. When near the vulture the cheel usually made one or two flaps. The black vulture responded by dipping movements of both wing-tips. Occasionally it made a large momentary canting resembling a strong depression of one wing. On reaching a height of about 500 metres the black vulture slow-flex-glided away, being chased for about half a mile by the cheel, which it slowly outdistanced. The teasing had gone on for about 10 minutes.

Thus, in this case, when the air was fully soarable the bird of heavy loading was able to glide horizontally at a greater speed than the bird of light loading.

Soaring and Non-soaring Dragon-flies.

But perhaps the most interesting case of soaring and non-soaring animals flying in the same air is afforded by the habits of the soaring dragon fly, Pantala flavescens, which in Calcutta often flies in company with another species of dragon fly, Rhyothemis variegata. This latter species has a restricted power of soaring, but is usually seen in flapping flight. These flapping dragon flies frequently fly together in a group consisting of from a few dozen to a few hundred individuals. The group remains for hours together in the same spot, often to leeward of a large tree. The individuals in the group keep rather close together so that any one dragon fly is rarely more than two or three feet distant from its nearest neighbours. Occasionally a few specimens of the soaring dragon fly may be seen gliding in the group. The following description is quoted from my book, "Animal Flight," page 388:—

> "It is noticeable that the Pantalas when gliding in a group with Rhyothemis travel at a much greater speed than the latter. That is to say, in the same air and within a few inches of one another, one dragon fly uses flapping in its flight. The other dispenses with flapping, but nevertheless travels at perhaps double the speed. An upholder of the ascending current theory might devote much learned—and perhaps useless—casuistry to explaining why the swift-gliding Pantala finds these supposed ascending currents, while the flapping Rhyothemis so consistently avoids them. The facts of the case contradict any theory that seeks to explain the gliding of Pantala by means of pre-existing movements in the air. Supposing that Pantala gets energy from the air by meeting eddies, or from some mysterious aerial vibration, or from some turbulent movement of the air, why is it that Rhyothemis is unaffected thereby though a lighter insect?"

In view of the above observations it is obviously desirable to accumulate data that will enable us to compare the flying capacities of different kinds of flying animals.

The Flight-Speed of the Locust.

With regard to the speed of the locust, observations are very easily made. A locust when liberated usually flies away horizontally for a considerable distance in a straight line. It is therefore easy to time it over a measured course. I found in this way that the speed of flight of locusts, both in my laboratory and in still air out of doors in the early morning, is 4 metres per second.

Accurate data are yet lacking for making a comparison between the speed of the locust and that of other insects. As compared with most butterflies its flight appears to be slow. In one experiment I found that a soaring dragon fly—

Tramea burmeisteri—could fly upwards by flapping on a course making an angle of about 60 degrees with the horizon at 4 metres per second and that when it had reached a height of about 3 metres it glided horizontally without flapping at a distinctly greater speed.

The Wing Loading of the Locust and Other Insects.

The following table permits of a comparison between the loading of the locust and that of other insects:—

TABLE I. LOADING OF INSECTS.

ORDER.	NAME.	LOADING. LBS. PER SQ. FOOT.	KILO. PER SQ. METRE.	AUTHORITY.
ORTHOPTERA.				
	Locust	.125	.61	H
	Dragon flies—			
	Rhyothemis variegata	.025	.12	H
	Maiden dragon fly	.029	.14	L
	Pantala flavescens	.032	.16	H
	Small dragon fly	.033	.16	de L
	Common dragon fly	.046	.23	de L
	Hemianax ephippiger	.066	.32	H
	Flat-bellied dragon fly	.094	.46	L
NEUROPTERA.				
	Ant lion	.033	.16	H
DIPTERA.				
	Gnat	.020	.10	L
	Psychoda bengalensis	.024	.12	H
	Daddy long legs (Tipula)	.068	.33	de L
	House fly	.114	.56	L
LEPIDOPTERA.				
	Cabbage butterfly	.017	.083	L
	Argynnis childreni	.024	.12	H
	Papilio ravana	.028	.14	H
	Swallow tail butterfly	.063	.31	L
	Hawk moth (Chærocampa celerio)	.167	.82	H
	Sphinx moth	.214	1.05	L
COLEOPTERA.				
	Cockchafer	.195	.96	L
	Stag beetle (Lucanus cervus), female ...	.214	1.05	L
	,, ,, ,, male ...	.266	1.30	L
	Rhinoceros beetle	.318	1.56	L
	Sternocera nitidicollis, calculated on all four wings	.73	3.60	H
	,, hind wings only	1.34	6.57	H
HYMENOPTERA.				
	Bee	.19	.93	de L

NOTE.—Measurements marked L are from Lougheed ("Vehicles of the Air," p. 162). Those marked de L are based on data given by de Lucy quoted in the article on Flight in the "Encyclopædia Britannica." My own measurements are marked H.

An interesting point to notice in this list is that the beetle Sternocera has a loading comparable to that of a vulture. When in flight this beetle makes a remarkably loud humming noise. Its speed ahead is very slow. Hence, it is improbable that the fore wings or wing cases aid appreciably in support. Another reason for holding this view is that in flight they are held at a dihedrally up angle of about 45°. Consequently it is probably correct to regard the figure 1.34, which is calculated on the area of the hind wings alone as the actual loading. The beetle is so strongly built that I had to use a hammer to drive a pin through the specimen I was dissecting.

Data are as yet lacking to enable us to make a profitable comparison between the wings of different insects. On the present occasion I will briefly describe the wings of the locust.

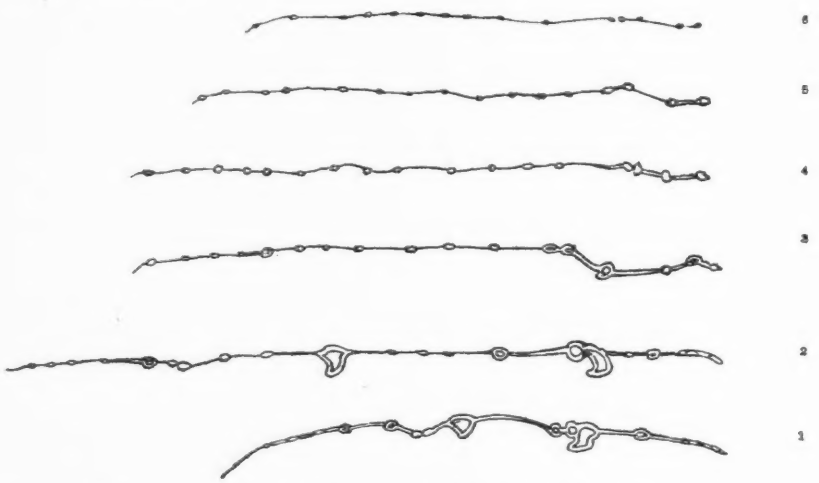

FIG. 1.—Transverse sections of fore wing of locust made at positions shown by lines numbered 1 to 6 in Fig. 3. The leading edge of each section is at the end near the number.

The Wings of the Locust.

A series of sections of the wings of the locust are shown in Figs. 1 and 2.

It will be noticed that the nervures stand out on both surfaces of the wing forming small ridges situated at right angles to the line of flight. The leading edge of the whole of the fore wing is very sharp, as is the leading edge of the outer half of the hind wing. The strength of the fore wing and the size of its nervures are probably partly due to its having to act as a protection to the hind wing when the insect is at rest.

The following illustration, Fig. 3, shows the outline of the Indian locust (Acridium peregrinum) with its wings extended. When the insect is at rest, as is the case with all grasshoppers, the hind wing is folded up like a fan and lies under the front wing. The hind wing of a grasshopper is shown expanded in Fig. 4 and folded up in Fig. 5.

When the wing is partly expanded the folds do not completely disappear and the wing membrane forms strong ridges that are partly situated at right

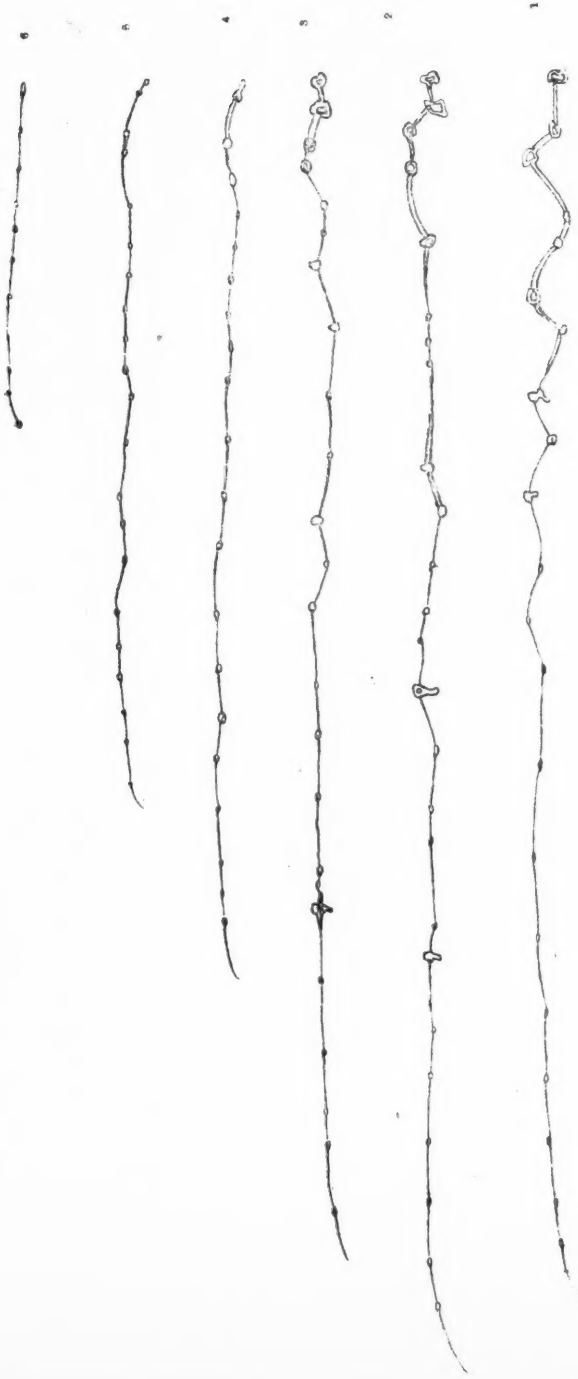

FIG. 2.—Transverse sections of hind wing of locust made at positions shown by lines numbered 1 to 6 in Fig. 3.

angles to the line of flight. It is a question of some interest how far these ridges disappear when the wing is fully expanded since I have shown elsewhere that the presence of ridges on the underside of the wing is characteristic of the more efficient soaring birds and other soaring animals ("Animal Flight," Chapter XIX.). In the case of a specimen of a locust that had been preserved in spirit I found that on unfolding the hind wing with my fingers the folds did not completely disappear. With an insect that had been kept in captivity till it was moribund and then killed the wing could be further flattened. In the case of an insect that was killed immediately when caught the unfolding could be carried further till the wing was completely flat except at the inner portion near its insertion on to the body. But to thus flatten the wing a certain amount of force was necessary and

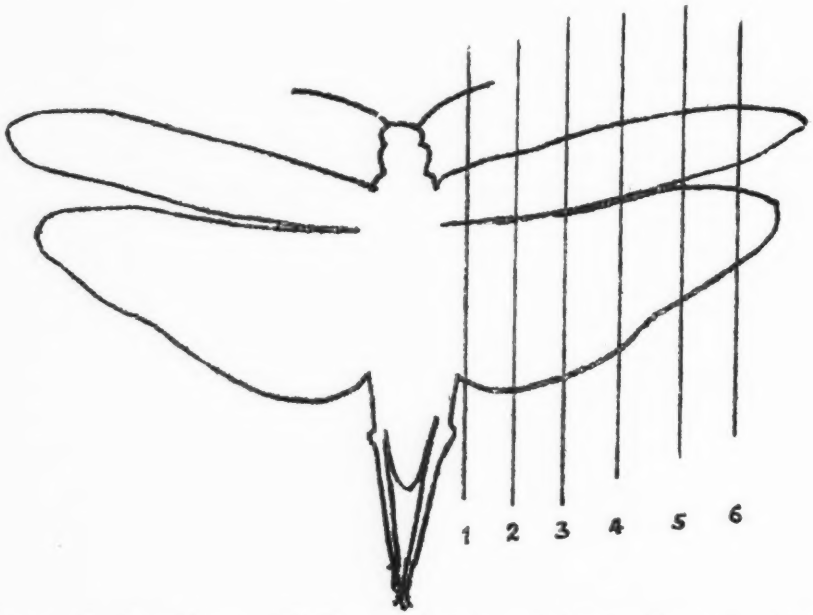

FIG. 3.—Outline of locust. The lines numbered 1 to 6 show positions of sections of wings shown in Figs. 4 and 5.

there is room for doubt how far the muscular effort of the insect could produce the same result. There can be no doubt that when flapping the outer part of the wing is perfectly free of folding. But it seems probable that the inner part retains slightly more folding than is indicated in the sections shown in Fig. 2.

The locust of which I have been treating is known in India as the north-west or migratory locust. It is capable of very long flights which all the evidence goes to show are entirely by flapping. Gliding appears to be used solely when the insect is descending to the ground. As I learn from an interesting communication from Mr. Maxwell-Lefroy, Entomologist to the Government of India, the swarms that reach India are "probably due to invasions from Arabia, Persia, Afghanistan, and Baluchistan, which enter India, breed (in desert portions of Western India), and the resulting swarms pass on eastwards to perish when they get beyond the sandy deserts in which alone they can exist permanently." A friend of mine

informs me that once, when out of sight of land on a P. and O. steamer in the Arabian Sea, he passed a swarm of locusts of such extent that it remained in sight on two successive days.

The Bombay locust (Acridium succinctum) has its permanent breeding place in forest areas in Western India, where it frequently forms swarms of small

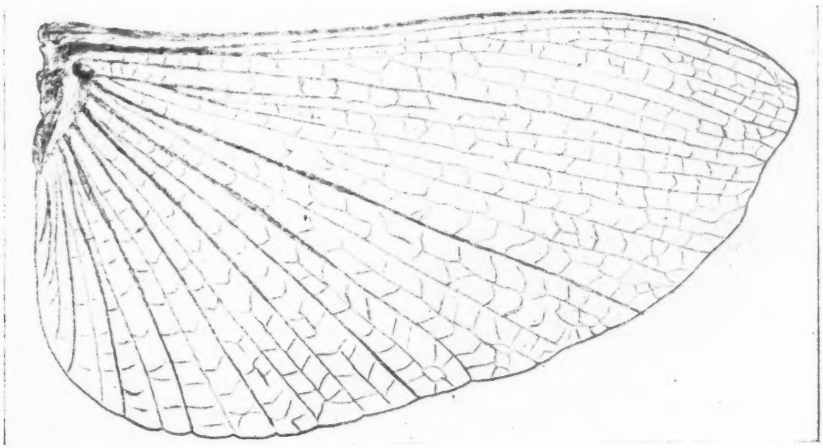

FIG. 4.—Hind wing of grasshopper (Pœcilocera picta) expanded.

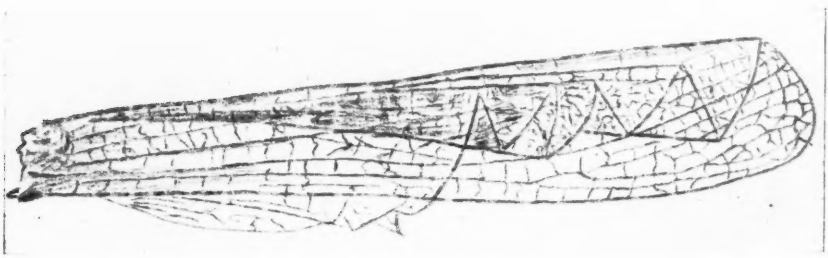

FIG. 5.—Hind wing of grasshopper (Pœcilocera picta) folded.

extent. Larger swarms occur at long intervals. In other parts of India it exists permanently in forest areas, but only in small numbers which never give rise to swarms. As a rule the limit of its flight by day is a few miles. It is stated to fly for longer distances at night.*

* See "The Bombay Locust," by H. Maxwell-Lefroy, published in "Memoirs of the Department of Agriculture in India, Entomological Series," Vol. I., No. 1, April, 1906, printed by Thacker, Spink and Co., Calcutta.

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